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ASYMPTOTIC SOLUTION OF CONTACT PROBLEMS FOR A RELATIVELY THICK ELASTIC LAYER WHEN THERE ARE FRICTION FORCES IN THE CONTACT AREA†

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The plane contact problem of the theory of elasticity of the interaction between a punch, having a base in the form of a paraboloid, and a layer, taking Coulomb friction in the contact region into account, is considered. It is assumed that either the lower boundary of the layer is fixed or there are no normal displacements and shear stresses on it, and that normal and shear forces are acting on the punch. Here, the punch-layer system is in a condition of limit equilibrium, and the punch does not turn during the deformation of the layer. The case of quasi-statistics, when the punch moves evenly over the layer surface, can be considered similarly in a moving system of coordinates. The problem is investigated by the large-λ method (see [1–3], etc.), which is further developed here, namely, simple recurrence relations are derived for constructing any number of terms of the series expansion of the solution of the corresponding integral equation in negative powers of the dimensionless parameter λ related to the thickness of the layer.

The aim of the present investigation was to obtain and analyse, by purely analytical methods, the results of a study of the effect of geometrical and mechanical parameters (especially Poisson's ratio and the thickness of the layer) on the position of the contact area), the shape of the deformed surface of the layer outside the contact area and the diagram of the contact stresses, taking into account the friction forces in the contact area. Earlier, these relations were investigated by the numerical solution of the integral equation for the problem of the interaction of a punch in the form of an elliptical paraboloid with an elastic layer [4].

Plane contact problems for a layer, taking into account the friction forces in the contact area, have been presented and investigated in many publications (see, for example, [2, 5, 6], etc.).

1. FORMULATION OF THE PROBLEM

We will consider an elastic layer $0 \le y \le h$ in Cartesian coordinates (x, y) (Fig. 1). Let a punch with a base in the form of a parabola with radius of curvature R at the vertex interact with the boundary of a layer $y = h$. A normal force P and a shear force $T = \mu P$ act on the punch, and forces of Coulomb friction with a coefficient of friction μ act in the contact area. Here, either the lower boundary of the layer $y = 0$ is fixed (Problem 1) or there are no normal displacements and shear stresses on it (Problem 2). The case of limit equilibrium is examined, and the punch does not turn during deformation of the layer.

By means of a Fourier transformation, the contact problems can be reduced, in terms of the unknown normal contact stresses beneath the punch $q(x)$ [6], to the following integral equation

$$\int_{-a}^{b} q(\xi)k\left(\frac{\xi-x}{h}\right)d\xi = \pi\theta\delta(x), \quad -a \le x \le b \tag{1.1}$$

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the kernel of which can be represented in the form of two terms

$$k(t) = k_1(t) - \varepsilon k_2(t), \quad \varepsilon = \frac{1-2\nu}{2(1-\nu)}\mu$$
$$k_1(t) = \int_0^\infty \frac{L_1(u)}{u}\cos ut du, \quad k_2(t) = \int_0^\infty \frac{L_2(u)}{u}\sin ut du \tag{1.2}$$

Here, for Problem 1

$$L_1(u) = [2\kappa \operatorname{sh} 2u - 4u]/\Delta^{(1)}(u)$$
$$L_2(u) = [2\kappa(\operatorname{ch} 2u - 1) - 4u^2(1-2\nu)^{-1}]/\Delta^{(1)}(u) \tag{1.3}$$
$$\Delta^{(1)}(u) = 2\kappa \operatorname{ch} 2u + 4u^2 + 1 + \kappa^2$$

and for Problem 2

$$L_1(u) = [\operatorname{ch} 2u - 1]/\Delta^{(2)}(u), \quad L_2(u) = [\operatorname{sh} 2u - 2(1-2\nu)^{-1}]/\Delta^{(2)}(u)$$
$$\Delta^{(2)}(u) = \operatorname{sh} 2u + 2u \tag{1.4}$$

We have introduced the following notation

$$\theta = \frac{G}{1-\nu}, \quad \delta(x) = \delta_0 - \beta x^2, \quad \beta = \frac{1}{2R}, \quad \kappa = 3 - 4\nu \tag{1.5}$$

where G is the shear modulus, ν is Poisson's ratio, μ is the coefficient of friction and δ_0 is the displacement of the punch in the vertical direction.

We will assume that the contact area $-a \le x \le b$ is not known in advance and depends on the magnitude of the force P.

Having replaced the variables

$$x = \eta(t), \quad \xi = \eta(\tau), \quad \eta(t) = \frac{a+b}{2}t - \frac{a-b}{2} \tag{1.6}$$

in Eq. (1.1), we convert this equation to the form

$$\int_{-1}^{1} \varphi(\tau) k\left(\frac{\tau - t}{\lambda}\right) d\tau = \pi f(t), \quad |x| \le 1 \tag{1.7}$$

where

$$\varphi(t) = q(\eta(t)), \quad f(t) = \frac{2\theta}{a+b}\delta(\eta(t)), \quad \lambda = \frac{2h}{a+b} \tag{1.8}$$

2. SOLUTION OF THE INTEGRAL EQUATION

To solve Eqs (1.7) and (1.8) with kernels (1.2)–(1.4), we will use the large-λ method. We will first convert the kernels (1.2)–(1.4) to the form [2]

$$k_1(t) = -\ln|t| + F_1(t), \quad k_2(t) = \frac{\pi}{2}\operatorname{sgn}(t) + F_2(t) \tag{2.1}$$

where the functions $F_i(t)$ can be represented in the form of the series

$$F_1(t) = -\sum_{i=0}^{\infty} d_i t^{2i}, \quad F_2(t) = \sum_{i=1}^{\infty} b_i t^{2i-1} \tag{2.2}$$

with coefficients

$$\begin{aligned} d_0 &= \int_0^{\infty} \frac{1 - L_1(u) - e^{-u}}{u} du, \quad d_i = \frac{(-1)^i}{(2i)!}\int_0^{\infty}[1 - L_1(u)]u^{2i-1}du, \quad i \geq 1 \\ b_i &= \frac{(-1)^i}{(2i-1)!}\int_0^{\infty}[1 - L_2(u)]u^{2i-2}du, \quad i \geq 1 \end{aligned} \tag{2.3}$$

It was shown in [2] that integral equations (1.7) and (2.1) are equivalent to the integral equation

$$\varphi(x) = \frac{\varepsilon_1}{\pi X(x)}\left\{P^* - \varepsilon_1 \int_{-1}^{1} \frac{f'(t)X(t)dt}{t-x} + \frac{\varepsilon_1}{\pi\lambda}\int_{-1}^{1}\frac{X(t)dt}{t-x}\int_{-1}^{1}\varphi(\xi)F'\left(\frac{\xi-t}{\lambda}\right)d\xi\right\} +$$
$$+ \varepsilon_0 f'(x) - \frac{\varepsilon_0}{\pi\lambda}\int_{-1}^{1}\varphi(\xi)F'\left(\frac{\xi-x}{\lambda}\right)d\xi \tag{2.4}$$

provided that

$$P^* = \int_{-1}^{1}\varphi(t)dt = \frac{\varepsilon_0}{\ln\lambda + D}\left\{\int_{-1}^{1}\frac{f(t)dt}{X(-t)} + \Phi\right\} \tag{2.5}$$

where

$$\Phi = \frac{1}{\pi}\int_{-1}^{1}\frac{dt}{X(-t)}\int_{-1}^{1}\varphi(\xi)F\left(\frac{\xi-t}{\lambda}\right)d\xi \tag{2.6}$$

Here, the following notation is used

$$F(t) = -F_1(t) + \varepsilon F_2(t), \quad X(x) = (1+x)^{1/2+\gamma}(1-x)^{1/2-\gamma}$$
$$D = -(\ln 2 + C + \psi(1/2+\gamma)/2 + \psi(1/2-\gamma)/2)$$
$$\gamma = \pi^{-1}\mathrm{arctg}\,\varepsilon, \quad \varepsilon_1 = 1/\sqrt{1+\varepsilon^2}, \quad \varepsilon_0 = \varepsilon/(1+\varepsilon^2)$$

where C is Euler's constant and $\psi(x)$ is Euler's ψ-function.

We will present the solution of Eq. (2.4) in the form of the expansion

$$\varphi(x) = \sum_{n=0}^{\infty}\lambda^{-n}\varphi_n(x) \tag{2.7}$$

and substitute it into the left- and right-hand sides of equality (2.4). After some reduction on the right-hand side, we equate the expressions on the left and right with like powers of λ. As a result we obtain the following recurrence relations for finding the functions $\varphi_n(x)$

$$\varphi_0(x) = \frac{\varepsilon_1}{\pi X(x)}[P^* - \varepsilon_1 f_0(x)] + \varepsilon_0 f'(x) \tag{2.8}$$

$$\varphi_n(x) = \frac{\varepsilon_1^2}{\pi^2 X(x)}\sum_{i=1}^{n} ic_i\int_{-1}^{1}\frac{X(\tau)d\tau}{\tau-x}\int_{-1}^{1}\varphi_{n-i}(t)(t-\tau)^{i-1}dt-$$

$$-\frac{\varepsilon_0}{\pi}\sum_{i=1}^{n} ic_i\int_{-1}^{1}\varphi_{n-i}(t)(t-x)^{i-1}dt \tag{2.9}$$

where c_i are the coefficients of the representation of the function $F(t)$ in the form of a series

$$F(t) = \sum_{i=0}^{\infty} c_i t^i, \quad c_{2i} = d_i, \quad c_{2i-1} = \varepsilon b_i \tag{2.10}$$

$$f_0(x) = \int_{-1}^{1}\frac{f'(\tau)X(\tau)}{\tau-x}d\tau \tag{2.11}$$

Subsequently, the values of the following integrals will be required [7, 8]

$$Q^* = \int_{-1}^{1}\tau^k X(\tau)d\tau = \frac{\pi}{2}(-1)^k(1-4\gamma)\sqrt{1+\varepsilon^2}F(3/2+\gamma,-k;3;2)$$

$$Q = \int_{-1}^{1}\frac{\tau^k d\tau}{X(\tau)} = \pi(-1)^k\sqrt{1+\varepsilon^2}F(1/2-\gamma,-k;1;2)$$

$$R_m^*(t) = \int_{-1}^{1}\frac{\tau^k X(\tau)d\tau}{\tau-t} = \pi\varepsilon t^m X(t)-\pi\sqrt{1+\varepsilon^2}t^m(2\gamma+t)+r_m(t) \tag{2.12}$$

$$r_m(t) = \sum_{k=0}^{m-1} Q_{m-k-1}^* t^k, \quad m \geq 1, \quad r_0(t) = 0$$

Here, $F(\alpha, -k; n; 2)$ is the hypergeometric function [7]. In the general case it can be represented as a hypergeometric series, but, since the second argument here is a negative integer, the series is terminated and transformed into a finite sum [7]

$$F(\alpha,-k;n;2) = \sum_{i=0}^{k-1}\frac{(\alpha)_i(-k)_i}{i!(n)_i}2^i, \quad (a)_i = a(a+1)\dots(a+i-1), \quad (a)_0 = 1$$

Consequently, Q_k and Q_k^* are also finite elementary sums.

Taking into account the fact that

$$f(t) = a_0+a_1t+a_2t^2, \quad f_0(t) = a_1R_0^*(t)+2a_2R_1^*$$

where

$$a_0 = \frac{2\theta}{a+b}\left(\delta_0-\beta\frac{(a-b)^2}{4}\right), \quad a_1 = \theta\beta(a-b), \quad a_2 = -\theta\beta\frac{a+b}{2} \tag{2.13}$$

we obtain

$$\varphi_0 = (\beta_{00}+\beta_{01}x+\beta_{02}x^2)/X(x)$$

$$\beta_{00} = \frac{\varepsilon_1 P^*}{\pi} + \theta\beta\varepsilon_1\left[2\gamma(a-b) + (1-4\gamma^2)\frac{a+b}{2}\right] \tag{2.14}$$

$$\beta_{01} = \theta\beta\varepsilon_1(a-b-2\gamma(a+b)), \quad \beta_{02} = -\theta\beta\varepsilon_1(a+b)$$

We will show that the function $X(x)\varphi_n(x)$ can be represented in the form of a polynomial of degree n

$$X(x)\varphi_n(x) = \sum_{k=0}^{n} \beta_{nk}x^k, \quad n \ge 1 \tag{2.15}$$

For this, in relations (2.9) we expand the binomial $(t-\tau)^{i-1}$, change the order of summation and integration and, after some lengthy reduction, obtain

$$\varphi_n(x) = \frac{1}{X(x)}\left\{\frac{\varepsilon_1^2}{\pi}\sum_{k=0}^{n-2} x^k \sum_{m=k+1}^{n-1} \frac{(-1)^m}{m!}\kappa_{nm}Q^*_{m-k-1} - \frac{\varepsilon_1}{\pi}(2\gamma+x)\sum_{m=0}^{n-1}\frac{(-1)^m x^m}{m!}\kappa_{nm}\right\} \tag{2.16}$$

where

$$\kappa_{nm} = \sum_{i=m+1}^{n} \frac{i!c_i}{(i-m-1)!}P^*_{n-i,\,i-m-1}, \quad P^*_{km} = \sum_{p=0}^{k} \beta_{kp}Q_{p+m} \tag{2.17}$$

In can be seen that the function $X(x)\varphi_n(x)$ is actually a polynomial of degree n. It can also be seen that, on the right-hand side of equality (2.16), the coefficients β_{kp} occur only when $k \le n-1$.

If expression (2.14) is now substituted into the left-hand side of equality (2.16) instead of $\varphi_n(x)$, then, to determine the coefficients β_{nk}, after some lengthy reduction we obtain the following recurrence relations

$$\begin{aligned}
\beta_{nn} &= \frac{\varepsilon_1}{\pi}(-1)^n c_n P_0^*, \quad n \ge 1 \\
\beta_{n,n-1} &= -\frac{2\gamma\varepsilon_1}{\pi}\frac{(-1)^{n-1}}{(n-1)!}r_{n,n-1} + \frac{\varepsilon_1(-1)^{n-1}}{\pi\,(n-2)!}r_{n,n-2}, \quad n \ge 1 \\
\beta_{nk} &= \frac{\varepsilon_1^2}{\pi^2}\sum_{m=k+1}^{n-1}\frac{(-1)^m}{m!}Q^*_{m-k-1}r_{nm} - \\
&-\frac{2\gamma\varepsilon_1}{\pi}\frac{(-1)^k}{k!}r_{nk} + \frac{\varepsilon_1}{\pi}\frac{(-1)^k}{(k-1)!}r_{n,k-1}, \quad 0 \le k \le n-2, \quad n \ge 2
\end{aligned} \tag{2.18}$$

Here

$$r_{nm} = \sum_{i=m+1}^{n-1} \frac{i!c_i}{(i-m-1)!}\sum_{p=0}^{n-i}\beta_{n-i,p}Q_{p+i-m-1} + \frac{n!c_n}{(n-m-1)!}\sum_{p=0}^{2}\beta_{0p}Q_{p+n-m-1} \tag{2.19}$$

Thus, the solution of Eq. (1.7) is finally represented in the form of (2.7), (2.14) and (2.15), taking into account recurrence relations (2.18).

Note that the recurrence relations (2.18) contain only arithmetic operations, which makes it possible to program them easily and, using programs for analytical transformations (of the MAPLE type), to obtain in analytical form any finite number of terms in expansions (2.7) and (2.15). This enables us to find a solution of the integral equation with any degree of accuracy in the region of convergence of series (2.7).

Below we will write out the coefficients β_{nk} only for the case when $n = 1$ and $n = 2$, although, in the numerical results given below, we will use coefficients β_{nk} with different $n > 2$, depending on the magnitude of the parameter λ and the specified accuracy

$$\beta_{10} = -\frac{2P^*}{\pi}\gamma\varepsilon\varepsilon_1 b_1, \quad \beta_{11} = -\frac{P^*}{\pi}\varepsilon\varepsilon_1 b_1$$

$$\beta_{20} = -\frac{P^*}{\pi}\varepsilon_1 d_1(1-12\gamma^2) + \theta\beta\gamma(1-4\gamma^2)\varepsilon_1 d_1\left[\frac{4}{3}\gamma(a+b) - 2(a-b)\right] \tag{2.20}$$

$$\beta_{21} = \frac{8P^*}{\pi}\varepsilon_1 d_1\gamma + \theta\beta(1-4\gamma^2)\varepsilon_1\left[\frac{2}{3}(a+b) - 2(a-b)\right], \quad \beta_{22} = \frac{2P^*}{\pi}d_1$$

3. DERIVATION OF THE PRINCIPAL RELATIONS

Knowing the solution of integral equation (1.7), we can find the contact stresses beneath the base of the punch at $-a \le x \le b$ from the formula

$$q(x) = \varphi\left(\frac{2x + a - b}{a + b}\right) \tag{3.1}$$

having first determined the boundaries of the contact area from the conditions for zero contact stresses at $x = -a$ and $x = b$. As a result we obtain the following system of two non-linear equations with the two unknowns a and b

$$\varphi(-1) = 0, \quad \varphi(1) = 0 \tag{3.2}$$

System (3.2) is cumbersome even if the solution of the integral equation is found apart from terms $O(\lambda^{-3})$. Therefore, in each specific case, the solution of system (3.2) will be found numerically with high accuracy, which does not present any great difficulties by virtue of the representation of the equations of the system in the form of polynomials in the required quantities equated to zero.

Note that, taking equalities (3.2) into account, the function $\varphi(t)$ can be represented in the form

$$\varphi(t) = X(-t)\varphi_*(t) \tag{3.3}$$

where $\varphi_*(t)$ is a continuous function bounded when $|t| \le 1$.

If the solution of Eq. (1.7) is found apart from terms $O(\lambda^{-2})$, then the formulae for determining the contact area asymptotically for high values of h ($h \gg \max(a, b)$) will take the form

$$a = a^- - \frac{d}{h} + O\left(\frac{1}{h^2}\right), \quad b = a^+ + \frac{d}{h} + O\left(\frac{1}{h^2}\right)$$
$$d = -\frac{P\mu b_1(1-2\nu)}{4\pi\beta G}, \quad a^\pm = \left[\frac{P(1-\nu)(1\mp 2\gamma)}{\pi\beta G(1\pm 2\gamma)}\right]^{1/2}, \quad 0 \le \gamma < \frac{1}{2} \tag{3.4}$$

In the case of Problem 2, the expression for d after evaluating the integral for b_1 by means of the last equation of (2.3) will be simplified and, for any values of ν, will take the form

$$d = \frac{P\mu b_1^*(1-\nu)}{2\pi\beta G}, \quad b_1^* = -\frac{1-2\nu}{2(1-\nu)}b_1 \approx 0.76857 \tag{3.5}$$

Note that in Problem 2 the quantity b_1^* does not depend on Poisson's ratio ν, while in Problem 1 it does depend on it.

In formulae (3.4) and (3.5), the quantity $d > 0$, and the interval $(-a^-, a^+)$ is the contact area when the punch interacts with the half-space, and here it is always the case that $a^- > a^+$, with the exception of the case of $\nu = 1/2$ ($\gamma = 0$), when $a^- = a^+$. On the basis of formulae (3.4) and (3.5) it is possible to draw certain preliminary conclusions concerning the nature of the dependence of the contact area on certain parameters: for example, when the thickness of the layer decreases, a decreases, while b increases. Furthermore, when the force P increases, d increases more rapidly than a^- and a^+. Thus, when the thickness of the layer decreases and the force P and Poisson's ratio ν increase, the contact area is displaced in the positive direction of the x axis, which has a considerable effect on the form of the distribution of the contact stresses, on the magnitude of their moment and on the deformation of the free surface. This will be discussed in greater detail below.

For numerical calculations it is necessary to find how the vertical displacement of the punch δ_0 depends on the applied normal force P and the shape of the deformed free surface outside the contact area.

To find δ_0, we will use relation (2.5) which, by means of expressions (2.13), we transform into

$$\delta_0 = \frac{a+b}{2\theta Q_0}\left\{(\ln\lambda + D)\frac{2P}{(a+b)\varepsilon_1} + \theta\beta\left[\frac{(a-b)^2}{2(a+b)}Q_0 + (a-b)Q_1 + \frac{a+b}{2}Q_2\right] - \Phi\right\}$$

We substitute the values of the function $\varphi(t)$ in the form (2.7), (2.15) into the expression for Φ (formula (2.6)). We obtain

$$\Phi = \frac{1}{\pi}\sum_{n=0}^{\infty}\lambda^{-n}\sum_{m=0}^{n}\frac{Q_m}{m!}S_{nm}$$

$$S_{nm} = \sum_{i=0}^{n-m}\frac{(n-i)!c_{n-i}}{(n-i-m)!}\sum_{p=0}^{i}\beta_{ip}Q_{n-i-m+p} + \frac{n!c_p}{(n-m)!}\sum_{p=1}^{2}\beta_{0p}Q_{n-m+p} \tag{3.6}$$

We will find the moment of the contact stresses. From the formula

$$M = \int_{-a}^{b} xq(x)dx$$

after replacement of the variables (1.6) we obtain

$$M = \frac{(a+b)^2}{4}\int_{-1}^{1} t\varphi(t)dt - \frac{a^2-b^2}{4}\int_{-1}^{1}\varphi(t)dt$$

Finally, in accordance with the representation of $\varphi(t)$ in the form of series (2.14), (2.15), we obtain

$$M = \frac{(a+b)^2}{4}\left[\sum_{n=1}^{\infty}\lambda^{-n}\sum_{k=0}^{n}\beta_{nk}Q_{k+1} + \sum_{k=0}^{2}\beta_{0k}Q_{k+1}\right] - \frac{a-b}{2}P \tag{3.7}$$

The displacement of the free surface outside the punch is represented by the relation

$$w(x) = W\left(\frac{2x+a-b}{a+b}\right), \quad x < -a, \quad x > b$$

$$W(t) = \frac{a+b}{2\pi\theta}\int_{-1}^{1}\varphi(\tau)k\left(\frac{\tau-t}{\lambda}\right)d\tau, \quad |t| > 1 \tag{3.8}$$

Taking relations (2.1) into account we have

$$W(t) = -\frac{a+b}{2\pi\theta}\left\{W_1(t) + \frac{\pi\varepsilon}{2}W_2(t) + W_3(t)\right\}$$

$$W_1(t) = \int_{-1}^{1} \varphi(\tau)\ln|\tau - t|d\tau - P^*\ln\lambda \tag{3.9}$$

$$W_2(t) = \int_{-1}^{1} \varphi(\tau)\operatorname{sgn}(\tau - t)d\tau, \quad W_3(t) = \int_{-1}^{1} \varphi(\tau)F\left(\frac{\tau - t}{\lambda}\right)d\tau$$

Using equalities (2.7) and (2.15), we obtain

$$W_1(t) = \sum_{k=0}^{N} \lambda^{-k} \sum_{m=0}^{k} \beta_{km}t_m(t) + \sum_{m=1}^{2} \beta_{0m}t_m(t) - \frac{2P\ln\lambda}{a+b} + O(\lambda^{-(N+1)})$$

$$t_m(t) = \int_{-1}^{1} \frac{\tau^m}{X(\tau)}\ln|\tau - t|d\tau \tag{3.10}$$

$$W_2(t) = \begin{cases} P^*, & \text{if} \quad t < -1 \\ -P^*, & \text{if} \quad t > 1 \end{cases}, \quad W_3(t) = \sum_{k=0}^{N} \lambda^{-k} \sum_{m=0}^{k} (-1)^m \frac{t^m}{m!} S_{mk} + O(\lambda^{-(N+1)})$$

4. NUMERICAL CALCULATIONS

Calculations were carried out to determine the boundaries of the contact area a and b, the contact stresses $q(x)$ $(-a \le x \le b)$, the moment of the contact stresses M, the vertical displacements of the punch δ_0 and the vertical displacements of the surface outside the contact area $w(z)$.

We will introduce the notation

$$q^*(x) = q(x)/G, \quad M^* = M/G, \quad \delta^* = \delta_0 G \tag{4.1}$$

Table 1 gives the values of $a, b, x_*, q^*(x_*), M^*$ and δ^* for Problems 1 and 2, where $x = x_*$ is the point of the contact area with the maximum contact stresses for certain values of the parameters $P_0 = P/G$, h, ν, μ and $R = 1$. The initial parameters were specified in the SI system. All results are given apart from terms $O(\lambda^{-7})$, besides the results of the first line of Table 1 (marked with an asterisk) which are calculated apart from terms $O(\lambda^{-11})$ and given in order to demonstrate the accuracy of the calculations.

Note that, if the contact area is calculated using the simplest asymptotic formulae (3.4), then, for $P_0 = 1$ and $h = 3$, for Problem 2 we find that $a = 0.834$ and $b = 0.695$ if $\nu = 0.1$, and $a = 0.587$ and $b = 0.597$ if $\nu = 0.45$. These results are similar to the corresponding results given in Table 1.

Figure 1 shows a graph of the distribution of the dimensionless contact stresses $q^*(x)$ in the contact area $-a \le x \le b$ for Problem 2 when $h = 1.0$, $\mu = 0.5$, $\nu = 0.1$ and $\nu = 0.45$.

The results of calculations given in Table 1 and Fig. 1 and also the simplest asymptotic formulae (3.4) enable us to draw a number of fundamentally important conclusions: when the thickness of the layer

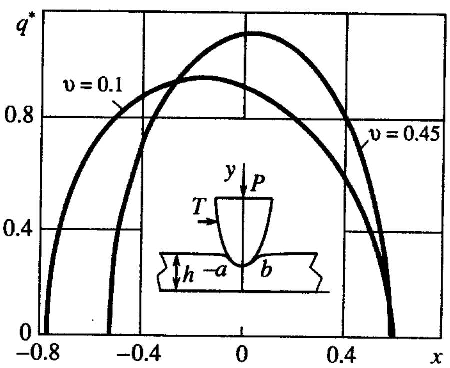

Fig. 1

Table 1

P_0	h	ν	$a \cdot 10^3$	$b \cdot 10^3$	$x_* \cdot 10^3$	$q^* \cdot 10^3$	$M^* \cdot 10^3$	$\delta^* \cdot 10^3$
		Problem 1, $\mu = 0.5$						
1	1	0.1	775*	578*	−186*	956*	−130*	362*
1	1	0.1	768	574	−188	962	−129	364
		0.3	638	552	−89.7	1077	−60.3	278
		0.45	513	540	9.16	1214	10.5	191
1	2	0.1	820	653	−182	873	−117	527
		0.3	687	607	−9.55	988	−59.2	406
		0.45	567	575	−9.34	1116	−.579	290
		Problem 2, $\mu = 0.5$						
1	1	0.1	764	599	−169	947	−114	369
		0.3	632	591	−64.8	1046	−36.4	307
		0.45	517	584	27.3	1159	30.2	259
1	2	0.1	816	663	−176	869	−110	537
		0.3	683	623	−86.0	976	−48.9	441
		0.45	567	592	−2.03	1099	7.66	366
1	3	0.1	834	672	−184	852	−116	647
		0.3	699	625	−96.1	964	−57.0	528
		0.45	582	590	−1.25	1086	−16.9	434
1/2	1	0.1	597	475	−135	597	−42.9	372
		0.3	501	441	−72.7	677	−22.2	302
		0.45	418	415	−13.3	764	−2.68	247
		Problem 2, $\mu = 0.9$						
1	1	0.1	826	544	−291	955	−196	366
		0.3	646	577	−113	1050	−63.1	304
		0.45	489	609	−49.2	1160	54.4	260
1	3	0.1	912	625	−327	846	−206	657
		0.3	734	602	−172	961	−102	530
		0.45	580	594	−22.5	1086	−3.01	434

h is reduced, either when the force P increases or when Poisson's ratio increases, the contact area is displaced in the positive direction of the x axis; when Poisson's ratio changes in the range 0–0.5, the moment of the contact stresses may change sign, and here the sign of the quantity x_*, defining the point of the contact area $x = x_*$ where the contact stresses are a maximum, may also change. Depending on the magnitudes of a and b, the nature of the deformation of the free surface in the vicinity of the punch also changes: if $a > b$ (the quantity ν is small), then, in the vicinity of the point $x = b$, the deformation is greater than in the vicinity of the point $x = -a$; if $a < b$ (the quantity ν is close to 0.5), then the opposite is the case. A value of Poisson's ratio ν will always be found where the pattern of the distribution of the contact stresses and the deformation of the free surface will be almost symmetrical, while the moment of the contact stresses will be zero. Furthermore, the displacement of the punch δ_0 is practically independent of the coefficient of friction μ.

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